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IMPROVEMENT OF THE CONVERGENCE OF FOURIER–HANKEL SERIES IN SOLVING TWO-DIMENSIONAL HEAT-CONDUCTION PROBLEMS

Yu. A. Kirsanov

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In the present work, an analytical solution of a boundary-value problem of nonstationary heat conduction in a two-dimensional solid (a prism, a cylinder) with improved convergence of Fourier–Hankel series is given. Using as an example a problem with cyclic inhomogeneous boundary conditions of the third kind, the author shows the uniformity of the convergence of the obtained solution on the boundaries of the solid and the matching of temperatures in passage from one period of the cycle to another.

Analytical solutions of heat-conduction problems with inhomogeneous boundary conditions are characterized by a drawback that is attributable to nonuniform convergence of series on the boundary of a solid body. The reason is that series cannot satisfy inhomogeneous boundary conditions [1]. Therefore, these solutions give an exact value only for the temperature averaged over the body volume; the temperature of the body surface is evaluated approximately. Since the solution of many problems requires a knowledge of precisely the temperature of the body surfaces, improvement of the convergence of Fourier–Hankel series is of great practical importance.

In dimensionless variables the boundary-value problem of heat conduction of the third kind for a two-dimensional body has the form

$$\frac{\partial\theta(x, y, \mathrm{Fo})}{\partial \mathrm{Fo}} = \frac{1}{x^{\nu}}\frac{\partial}{\partial x}\left[x^{\nu}\frac{\partial\theta(x, y, \mathrm{Fo})}{\partial x}\right] + \frac{1}{L^2}\frac{\partial^2\theta(x, y, \mathrm{Fo})}{\partial y^2}\,; \tag{1}$$

$$\theta(x, y, 0) = \psi(x, y, 0)\,; \tag{2}$$

$$\frac{\partial\theta(0, y, \mathrm{Fo})}{\partial x} = 0\,; \tag{3}$$

$$\frac{\partial\theta(1, y, \mathrm{Fo})}{\partial x} = -\mathrm{Bi_w}\left[\theta(1, y, \mathrm{Fo}) - \theta_{\mathrm{f}}(y, \mathrm{Fo})\right]; \tag{4}$$

$$\frac{\partial\theta(x, 0, \mathrm{Fo})}{\partial y} = L\,\mathrm{Bi}_0\left[\theta(x, 0, \mathrm{Fo}) - \theta_{\mathrm{f}}(0, \mathrm{Fo})\right]; \tag{5}$$

$$\frac{\partial\theta(x, 1, \mathrm{Fo})}{\partial y} = -L\,\mathrm{Bi}_1\left[\theta(x, 1, \mathrm{Fo}) - \theta_{\mathrm{f}}(1, \mathrm{Fo})\right]. \tag{6}$$

The analytical solution of problem (1)-(6) is an expansion in Fourier–Hankel series:

Kazan State Power-Engineering Institute, Russia; e-mail: ikv@kspei.kcn.ru. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 73, No. 6, pp. 1352-1357, November–December, 2000. Original article submitted July 20, 1999.

$$\theta(x, y, \mathrm{Fo}) = \sum_{m=1}^{\infty} \frac{x^{\nu} K_x(\mu_m x)}{S_x} \sum_{n=1}^{\infty} \frac{K_y(\gamma_n y)}{S_y} f(\mu_m, \gamma_n, \mathrm{Fo}) . \tag{7}$$

Here

$$K_x(\mu_m x) = \begin{cases} \cos(\mu_m x) & \text{for } \nu = 0 , \\ J_0(\mu_m x) & \text{for } \nu = 1 ; \end{cases}$$

$$K_y(\gamma_n y) = \cos(\gamma_n y) + L \frac{\mathrm{Bi}_0}{\gamma_n} \sin(\gamma_n y) ;$$

μ_m are the roots of the characteristic equation

$$\mathrm{Bi_w} = \mu \tan(\mu) \quad \text{for} \quad \nu = 0 , \quad \mathrm{Bi_w} J_0(\mu) = \mu J_1(\mu) \quad \text{for} \quad \nu = 1 ;$$

γ_n are the roots of the characteristic equation

$$\tan(\gamma)(\gamma/L - L\, \mathrm{Bi}_0 \mathrm{Bi}_1/\gamma) = \mathrm{Bi}_0 + \mathrm{Bi}_1 ;$$

$$f(\mu_m, \gamma_n, \mathrm{Fo}) = F(\mathrm{Fo}) + \exp\left[-\left(\mu_m^2 + \gamma_n^2/L^2\right)\right] \theta_{\mathrm{L}}(\mu_m, \gamma_n, 0) ;$$

$$F(\mathrm{Fo}) = \exp\left[-\left(\mu_m^2 + \gamma_n^2/L^2\right)\mathrm{Fo}\right] \int_0^{\mathrm{Fo}} W(\mathrm{Fo}) \exp\left[\left(\mu_m^2 + \gamma_n^2/L^2\right)\eta\right] d\eta ;$$

$$W(\mathrm{Fo}) = \mathrm{Bi_w} K_x(\mu_m) \int_0^1 K_y(\gamma_n y)\, \theta_{\mathrm{f}}(y, \mathrm{Fo})\, dy +$$

$$+ \left[\mathrm{Bi}_0\, \theta_{\mathrm{f}}(0, \mathrm{Fo}) + \mathrm{Bi}_1 K_y(\gamma_n)\, \theta_{\mathrm{f}}(1, \mathrm{Fo})\right] \int_0^1 x^{\nu} K_x(\mu_m x)\, dx ;$$

$$S_x = \int_0^1 x^{\nu} K_x^2(\mu_m x)\, dx ; \quad S_y = \int_0^1 K_y^2(\gamma_n y)\, dy .$$

The temperature averaged over the body volume is

$$\bar{\bar{\theta}}(\mathrm{Fo}) = \sum_{m=1}^{\infty} \frac{\overline{K_x}(\mu_m)}{S_x} \sum_{n=1}^{\infty} \frac{\overline{K_y}(\gamma_n)}{S_y} f(\mu_m, \gamma_n, \mathrm{Fo}) , \tag{8}$$

where

$$\overline{K_x}(\mu_m) = \int_0^1 x^{\nu} K_x(\mu_m x)\, dx ; \quad \overline{K_y}(\gamma_n) = \int_0^1 K_y(\gamma_n y)\, dy .$$

Existing methods for improving the convergence of Fourier–Hankel series [2-4] have shown good performance in one-dimensional problems. In the present work, an attempt is made to extend one of these methods [4] to two-dimensional problems. For this, to solution (7) we add a certain function $V^*(x, y, \mathrm{Fo})$ and simultaneously subtract the function $V(x, y, \mathrm{Fo})$. Both functions are solutions of the quasistationary problem

$$\frac{1}{x^\nu}\frac{\partial}{\partial x}\left[x^\nu\frac{\partial V(x, y, \mathrm{Fo})}{\partial x}\right]+\frac{1}{L^2}\frac{\partial^2 V(x, y, \mathrm{Fo})}{\partial y^2}=0\,; \tag{9}$$

$$\frac{\partial V(0, y, \mathrm{Fo})}{\partial x}=0\,; \tag{10}$$

$$\frac{\partial V(1, y, \mathrm{Fo})}{\partial x}=-\,\mathrm{Bi_w}\,[V(1, y, \mathrm{Fo})-\theta_\mathrm{f}(y, \mathrm{Fo})]\,; \tag{11}$$

$$\frac{\partial V(x, 0, \mathrm{Fo})}{\partial y}=L\,\mathrm{Bi}_0\,[V(x, 0, \mathrm{Fo})-\theta_\mathrm{f}(0, \mathrm{Fo})]\,; \tag{12}$$

$$\frac{\partial V(x, 1, \mathrm{Fo})}{\partial y}=-\,L\,\mathrm{Bi}_1\,[V(x, 1, \mathrm{Fo})-\theta_\mathrm{f}(1, \mathrm{Fo})]\,. \tag{13}$$

The function $V(x, y, \mathrm{Fo})$ is the solution of problem (9)-(13) in the form of the series

$$V(x, y, \mathrm{Fo})=\sum_{m=1}^{\infty}\frac{x^\nu K_x(\mu_m x)}{S_x}\sum_{n=1}^{\infty}\frac{K_y(\gamma_n y)}{S_y}\frac{W(\mathrm{Fo})}{\mu_m^2+\gamma_n^2/L^2}\,.$$

The function $V^*(x, y, \mathrm{Fo})$ is the solution of the same problem in closed form. Having represented it in the form of the product $V^* = X(x)Y(y)$, by the method of separation of variables we obtain

$$V^*(x, y, \mathrm{Fo})=\frac{\theta_\mathrm{f}(y, \mathrm{Fo})}{X(x)\,Y(y)}(A+By)\,,$$

where

$$A=\frac{\mathrm{Bi}_0\,\theta_\mathrm{f}(0, \mathrm{Fo})\,(1+L\,\mathrm{Bi}_1)+\mathrm{Bi}_1\theta_\mathrm{f}(1, \mathrm{Fo})}{\mathrm{Bi}_0+\mathrm{Bi}_1+L\,\mathrm{Bi}_0\mathrm{Bi}_1}\,;$$

$$B=\frac{L\,\mathrm{Bi}_0\mathrm{Bi}_1\,[\theta_\mathrm{f}(1, \mathrm{Fo})-\theta_\mathrm{f}(0, \mathrm{Fo})]}{\mathrm{Bi}_0+\mathrm{Bi}_1+L\,\mathrm{Bi}_0\mathrm{Bi}_1}\,.$$

From the solution obtained it follows that the quantity V^* is independent of x and formally it can be equal to one of the three expressions

$$V^*(y, \mathrm{Fo})=\sqrt{\theta_\mathrm{f}(y, \mathrm{Fo})\,(A+By)}\,, \tag{14.a}$$

$$V^*(y, \mathrm{Fo})=A+By\,, \tag{14.b}$$

$$V^*(y, \mathrm{Fo})=\theta_\mathrm{f}(y, \mathrm{Fo})\,. \tag{14.c}$$

Thus, the solution of initial problem (1)-(6) with the improved convergence of the series has the form

$$\theta\,(x, y, \mathrm{Fo}) = V^*\,(y, \mathrm{Fo}) +$$

$$+\sum_{m=1}^{\infty}\frac{x^{\nu}K_x\,(\mu_m x)}{S_x}\sum_{n=1}^{\infty}\frac{K_y\,(\gamma_n y)}{S_y}\left[f\,(\mu_m, \gamma_n, \mathrm{Fo})-\frac{W\,(\mathrm{Fo})}{\mu_m^2+\gamma_n^2/L^2}\right]. \tag{15}$$

The body-volume-averaged temperature for the solution with the improved convergence is as follows:

$$\bar{\bar{\theta}}\,(\mathrm{Fo}) = \int_0^1 V^*\,(y, \mathrm{Fo})\,dy +$$

$$+\sum_{m=1}^{\infty}\frac{\overline{K_x}\,(\mu_m)}{S_x}\sum_{n=1}^{\infty}\frac{\overline{K_y}\,(\gamma_n)}{S_y}\left[f\,(\mu_m, \gamma_n, \mathrm{Fo})-\frac{W\,(\mathrm{Fo})}{\mu_m^2+\gamma_n^2/L^2}\right]. \tag{16}$$

Subsequent calculations showed that substitution of any expression of (14a)-(14c) into Eq. (15) gives a monotonic temperature distribution near the body boundaries, but the resulting values of the temperatures on the surface are different. Therefore, the question arose concerning the selection of a true expression for $V^*(y$, Fo), for which purpose we considered the problem of cyclic heat exchange between the body and some media. In this problem, initial condition (2) takes the form of a condition for the switching of periods [5, 6]:

$$\theta_{j+1}\,(x, y, 0) = \theta_j\,(x, y, \mathrm{Fo}_{T,j})\,. \tag{2.a}$$

The expression for $V^*(y$, Fo) was selected proceeding from two conditions: 1) equality of the average temperatures determined from formulas (8) and (16) and 2) satisfaction of the condition for switching periods (2a).

Solution of the problem of cyclic heat exchange presupposes determination of the representation of the initial temperature field $\theta_L(\mu_m, \gamma_n, 0)$ for the j-th period that enters into $f(\mu_m, \gamma_n$, Fo). This representation is found by means of the condition for switching periods (2a). For a two-period cycle, we can obtain (in subsequent expressions variable quantities will be indexed in conformity with the period number):

$$\frac{\theta_{\mathrm{L}}\,(\mu_{m_j}, \gamma_{n_j}, 0)}{\varphi_j} = U_j + \sum_{m_1=1}^{\infty}\frac{S_{x,j,j1}}{S_{x,j1,j1}}\sum_{n_1=1}^{\infty}\frac{S_{y,j,j1}}{S_{y,j1,j1}}\times$$

$$\times\left\{R_{j1}+\exp\left[-\left(\mu_{m_1}^2+\gamma_{n_1}^2/L^2\right)\mathrm{Fo}_{T,j1}\right]\times\right.$$

$$\times\left\langle U_{j1}+\sum_{m_2=1}^{\infty}\frac{S_{x,j1,j2}}{S_{x,j2,j2}}\sum_{n_2=1}^{\infty}\frac{S_{y,j1,j2}}{S_{y,j2,j2}}R_{j2}+\right.$$

$$+\sum_{\substack{m_2=1\\ m_2\neq m_j}}^{\infty}\frac{S_{x,j1,j2}}{S_{x,j2,j2}}\sum_{\substack{n_2=1\\ n_2\neq n_j}}^{\infty}\frac{S_{y,j1,j2}}{S_{y,j2,j2}}\exp\left[-\left(\mu_{m_2}^2+\gamma_{n_2}^2/L^2\right)\mathrm{Fo}_{T,j2}\right]\times$$

TABLE 1. Temperatures Averaged over the Prism Volume

Number of period	Fo	Calculations of $\overline{\overline{\theta}}_j(\mathrm{Fo})$ by formulas			
		(8)	(16) and (14.a)	(16) and (14.b)	(16) and (14.c)
1	0	1.134050	1.139651	1.136365	1.134074
	1	0.865914	0.867782	0.863615	0.865922
0	0	0.865914	0.861634	0.863635	0.865927
	1	1.134050	1.133074	1.136385	1.134078

$$\times \theta_{\mathrm{L}} (\mu_{m_2}, \gamma_{n_2}, 0)\Big\rangle\Bigg\} ;$$

$$U_k = \frac{W_k (0)}{\mu_{m_k}^2 + \gamma_{n_k}^2 / L^2} + D^*_{k,k-1} ; \quad R_k = F_k (\mathrm{Fo}_{T,k}) - \frac{W_k (\mathrm{Fo}_{T,k})}{\mu_{m_k}^2 + \gamma_{n_k}^2 / L^2} ;$$

$$\varphi_j = \Bigg\{ 1 - \frac{\exp\left[-\left(\mu_{m_j}^2 + \gamma_{n_j}^2 / L^2\right)\mathrm{Fo}_{T,j}\right]}{S_{x,j,j}\, S_{y,j,j}} \sum_{m_1=1}^{\infty} \frac{S_{x,j,j1}\, S_{x,j1,j}}{S_{x,j1,j1}} \times$$

$$\times \sum_{n_1=1}^{\infty} \frac{S_{y,j,j1}\, S_{y,j1,j}}{S_{y,j1,j1}} \exp\left[-\left(\mu_{m_1}^2 + \gamma_{n_1}^2 / L^2\right)\mathrm{Fo}_{T,j1}\right]\Bigg\}^{-1} ;$$

$$D^*_{i2,i1} = \int_0^1 K_x (\mu_{m_{i2}} x)\, dx \int_0^1 K_y (\gamma_{n_{i2}} y)\, [V^*_{i1} (y, \mathrm{Fo}_{T,i1}) - V^*_{i2} (y, 0)]\, dy ;$$

$$S_{x,\, i1,i2} = \int_0^1 x^{\nu} K_x (\mu_{m_{i1}} x)\, K_x (\mu_{m_{i2}} x)\, dx ; \quad S_{y,\, i1,i2} = \int_0^1 K_y (\gamma_{n_{i1}} y)\, K_y (\gamma_{n_{i2}} y)\, dy .$$

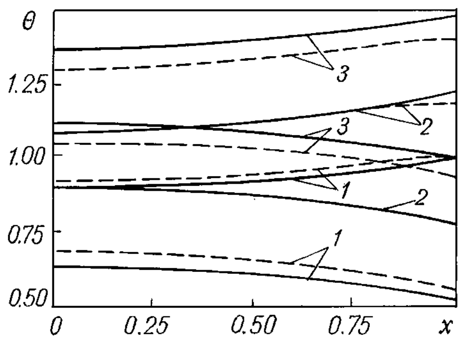
$$j1 = j + 1 ; \quad j2 = j + 2 ; \quad m_1 = m_{j1} ; \quad m_2 = m_{j2} ; \quad n_1 = n_{j1} ; \quad n_2 = n_{j2} .$$

For the sake of definiteness, we consider the given problem with a certain, for example, exponential, law of change in the heat-carrier temperature:

$$\theta_{\mathrm{f},j} (y, \mathrm{Fo}) = \theta_{0,j} + b_j \exp (-\beta_j \mathrm{Fo}) \left[1 + \frac{y_{0,j} \pm Ly}{a_j} - \exp\left(-\frac{y_{0,j} \pm Ly}{a_j}\right)\right] . \tag{17}$$

Here the "+" sign before L refers to the period of cooling of the body ($j = 1$), while the "–" sign refers to the period of heating of the body ($j = 2$). Let $\theta_{0,1} = 0$; $\theta_{0,2} = 2$; $\beta_1 = \beta_2 = 0.05$; $b_1 = -b_2 = 0.3$; $L = a_1 = a_2 = 16$; $\mathrm{Bi}_{\mathrm{w},1} = \mathrm{Bi}_{\mathrm{w},2} = 0.4$; $\mathrm{Bi}_{0,1} = \mathrm{Bi}_{0,2} = \mathrm{Bi}_{1,1} = \mathrm{Bi}_{1,2} = 0.05$; $\mathrm{Fo}_{T,1} = \mathrm{Fo}_{T,2} = 1$.

Substitution of formula (17) into the expression for $\theta_{\mathrm{L}}(\mu_{m_j}, \gamma_{n_j}, 0)$ allows us to evaluate $W(\mathrm{Fo})$, $F(\mathrm{Fo})$, and $f(\mu_{m_j}, \gamma_{n_j}, \mathrm{Fo})$ that enter into Eqs. (15) and (16) (for more detail, see [6]). Calculation of $D^*_{1,2}$ and $D^*_{2,1}$ was carried out by a numerical method from Simpson's formula over individual intervals whose boundaries were determined by the following values of the longitudinal coordinate y: 0, 0.01, 0.025, 0.05, 0.075, 0.1, 0.15, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.85, 0.9, 0.925, 0.95, 0.975, 0.99, and 1.



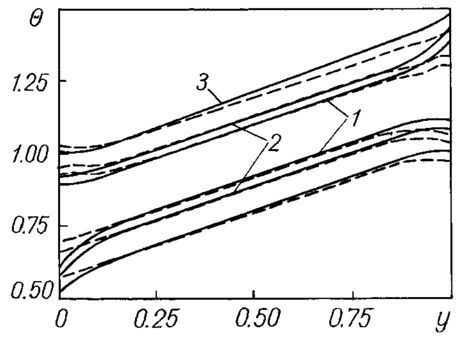


Fig. 1. Change in the temperature of the prism in the transverse direction: 1) on the front end (y = 0); 2) in the middle plane (y = 0.5); 3) on the back end (y = 1); dashed curves, according to Eq. (7); solid curves, according to Eq. (15).

Fig. 2. Change in the temperature of the prism in the longitudinal direction: 1) on the axis (x = 0); 2) in the depth (x = 0.5); 3) on the surface (x = 1); dashed curves, according to Eq. (7); solid curves, according to Eq. (15).

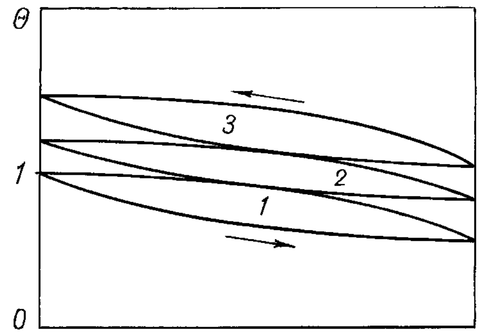


Fig. 3. Changes with time over a cycle of the surface temperatures (x = 1): 1) the leading edge (y = 0); 2) the middle of the prism (y = 0.5); 3) the trailing edge (y = 1).

Calculations of $\bar{\bar{\theta}}_j$(Fo) performed for a prism for $m_1 = m_2 = 6$ and $n_1 = n_2 = 10$, whose results are given in Table 1, indicate that use of function (14.c) in solving Eq. (16) provides virtually the same values of $\bar{\bar{\theta}}_j$ (Fo) as Eq. (8). Moreover, function (14.c) satisfies condition (2.a) more exactly than functions (14.a) and (14.b). Therefore, it can be concluded that function (14.c) is the most correct expression for $V_j^*(y, \text{Fo})$.

The influence of the improvement of the convergence of the Fourier–Hankel series on the temperature fields in the prism is shown in Figs. 1 and 2. Although the number of terms of the series in expression (7) was taken to be equal to $m_1 = m_2 = 10$ and $n_1 = n_2 = 20$, i.e., almost twice that in expression (15), the temperature fields near the boundaries of the body, according to solution (7), converge nonuniformly. This is especially noticeable for the period of heating of the body (j = 2).

The change, shown in Fig. 3 over a cycle of the temperatures of the prism surface indicate that not only the average temperature of the body but also the temperatures of individual points of its surface satisfy the condition of switching (2.a), whereas expressions (14.a) and (14.b) do not provide satisfaction of this condition.

CONCLUSIONS

1. A solution of a nonstationary problem of heat conduction in a two-dimensional solid body under inhomogeneous boundary conditions of the third kind with improved convergence of the Fourier–Hankel series is obtained.

2. Convergence of the solution on the boundaries of the body and satisfaction of the condition for switching the periods are shown using, as an example, two-period cyclic variation of the boundary conditions.

NOTATION

ν = 0 and 1, for a prism and a cylinder, respectively; j, number of the period; $\theta_j(x, y, \text{Fo})$, relative excess temperature of the body in the j-th period; $\theta_{f,j}(y, \text{Fo})$, relative excess temperature of the medium that washes the body; x, transverse coordinate of the body referred to its half-width, $0 < x < 1$; y, longitudinal coordinate of the body referred to its length, $0 < y < 1$; $\text{Fo} = a\tau/l^2$, Fourier number; $\text{Fo}_{T,j} = aT_j/l^2$, limiting Fourier number for the j-th period; a, coefficient of thermal diffusivity of the body, m^2/sec; l, body half-width, m; L, length of the body referred to its half-width; τ, time from the beginning of the j-th period, sec; T_j, duration of the j-th period, sec; $\text{Bi}_{w,j} = \alpha_{w,j}l/\lambda$, $\text{Bi}_{0,j} = \alpha_{0,j}l/\lambda$, and $\text{Bi}_{1,j} = \alpha_{1,j}l/\lambda$, Biot numbers for the side surface and the front and back ends of the body in the j-th period; $\alpha_{w,j}$, $\alpha_{0,j}$, and $\alpha_{1,j}$, coefficients of heat transfer of the side surface and the front and back ends of the body in the j-th period, W/(m^2·K); λ, coefficient of thermal conductivity of the body, W/(m·K); $J_r(z)$, first-kind Bessel function of r-th order. Subscripts: f, flux; L, dimensionless temperature of the body in the Fourier–Hankel transforms.

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